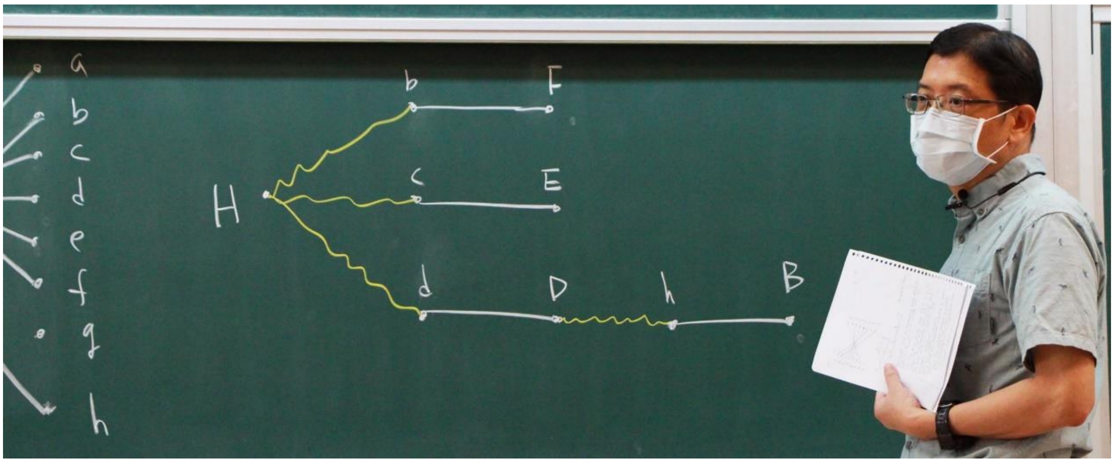
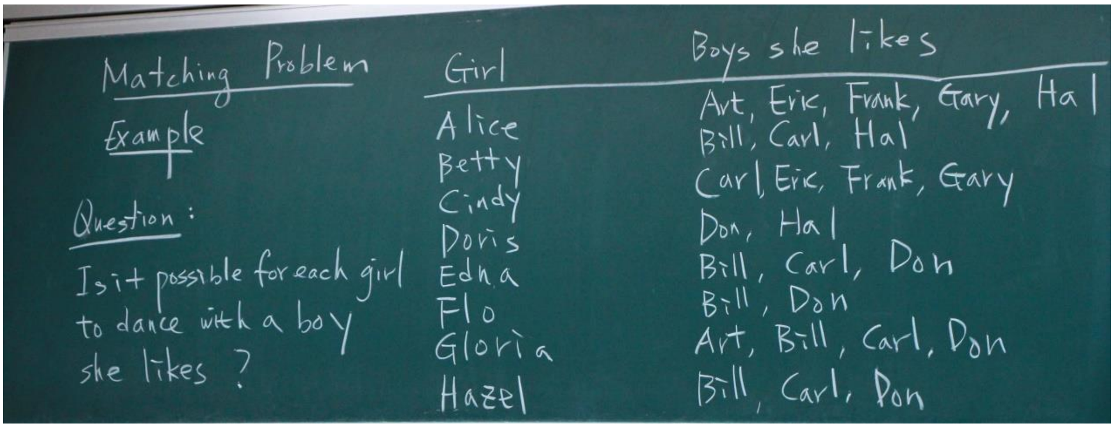
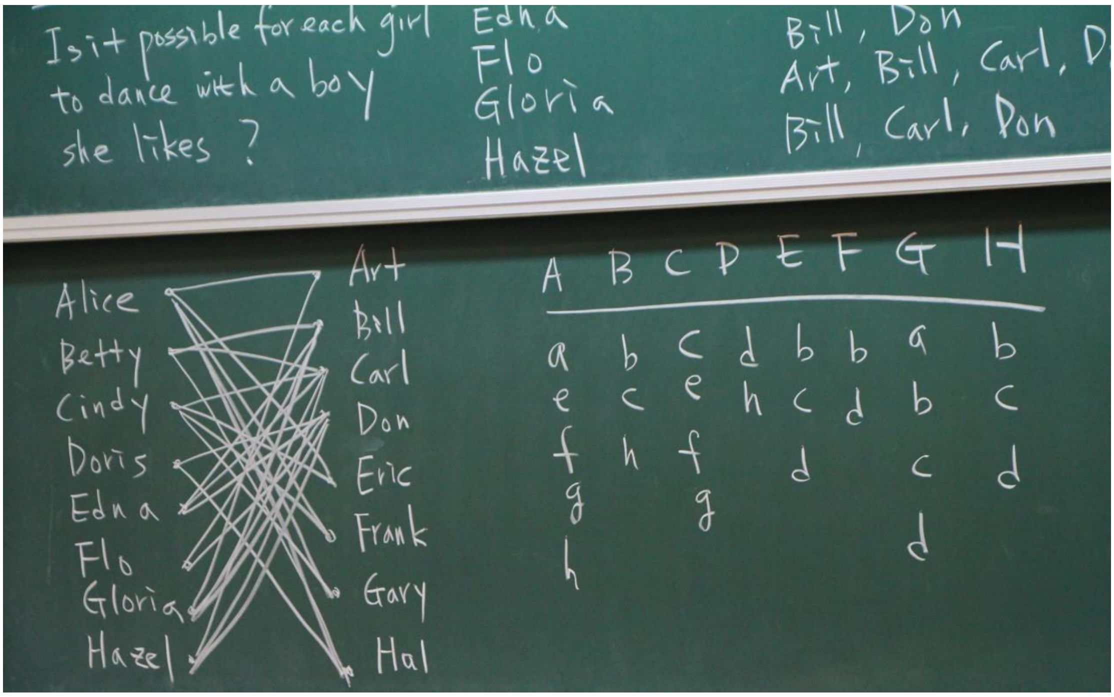
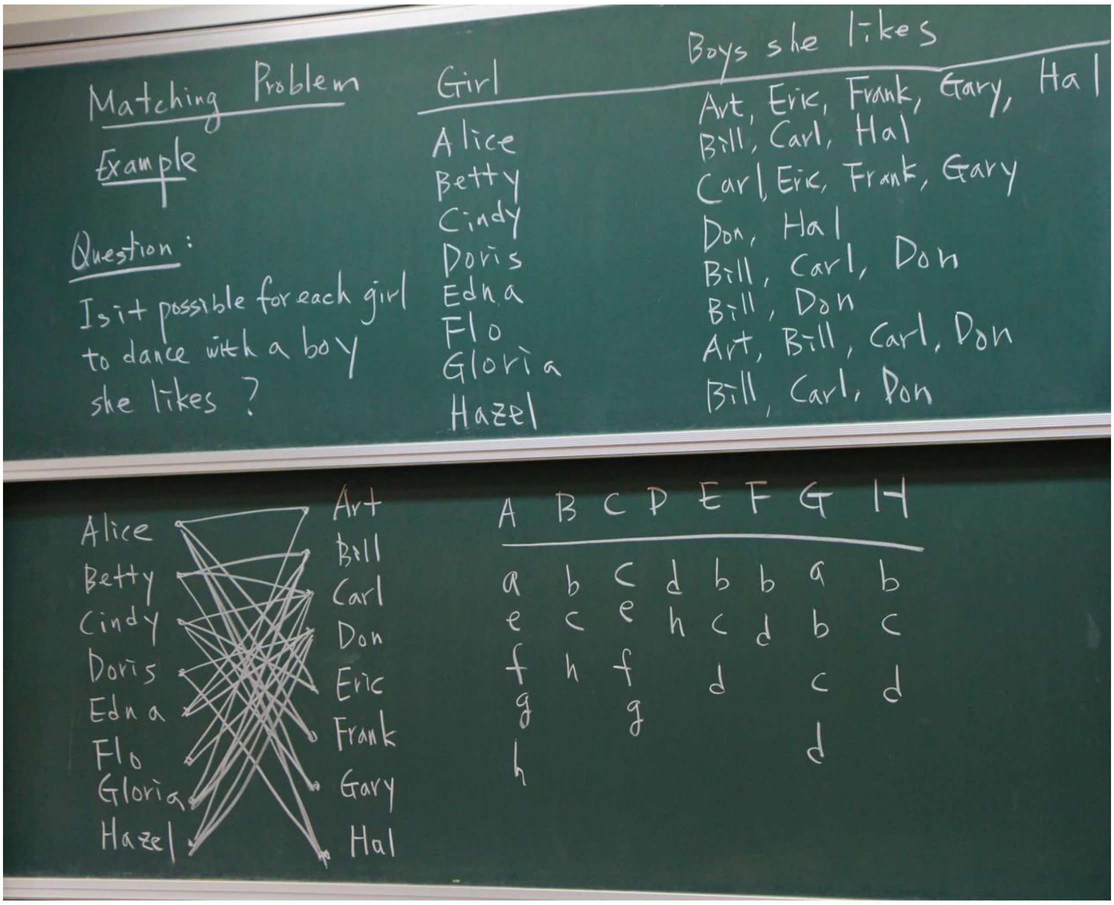
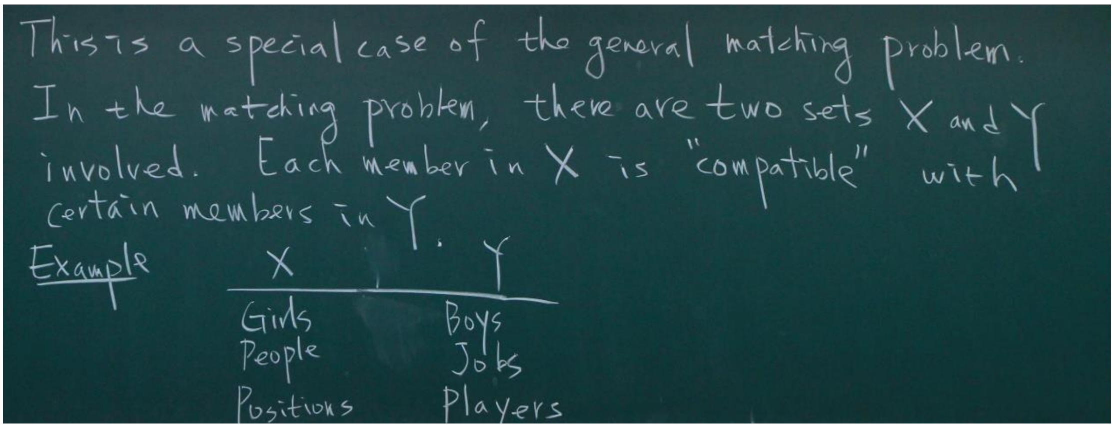
【10920 趙啟超教授離散數學 / 第 29 堂版書】








Matching Problem
Example
Question:
Is it possible for each girl to dance with a boy she likes?
Girl
Boys she likes
Alice
Art, Eric, Frank, Gary, Hal
Betty
Bill, Carl, Hal
Cindy
Carl, Eric, Frank, Gary
Doris
Don, Hal
Edna
Bill, Carl, Don
Flo
Bill, Don
Gloria
Art, Bill, Carl, Don
Hazel
Bill, Carl, Don
Alice
Betty
Cindy
Doris
Edna
Flo
Gloria
Hazel
Art
Bill
Carl
Don
Eric
Frank
Gary
Hal
A B C D E F G H
a b c d b b a b
e c e h c d b c
f h f d c d
g g d
h


This is a special case of the general matching problem.
In the matching problem, there are two sets X and Y involved. Each member in X is "compatible" with certain members in Y.
Example
X
Y
Girls
Boys
People
Jobs
Positions
Players

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<u>Example</u>

X	Y
Girls	Boys
People	Jobs
Positions	Players

A problem of this kind can be described by a bipartite graph.

<u>Def</u> Let $G=(V,E)$ be a partite graph with $V=X\cup Y$. (Each edge of E has one vertex in X and the other in Y.)

1. A matching is a subset M of E with the property that no two edges in M have a common vertex.

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Def Let $G=(V,E)$ be a partite graph with $V = X \cup Y$.
(Each edge of E has one vertex in X and the other in Y.)

1. A matching is a subset M of E with the property that no two edges in M have a common vertex.
2. A matching M is a maximal matching if no other matching has a greater cardinality.
3. A matching M is a complete matching if $|M| = |X|$.

Example (Continued)

Example (Continued)

A	B	C	D	E	F	G	H
a	b	c	d	b	b	a	b
e	c	e	h	c	d	b	c
f	h	f		d		c	d
g		g				d	
h							

A —— a
B —— b
C —— c
D —— d
E   e
F   f
G   g
H   h

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e	c	e	h	c	d	b	c
f	h	f		d		c	d
g		g				d	
h							

A —— a
B —— b
C —— c
D —— d
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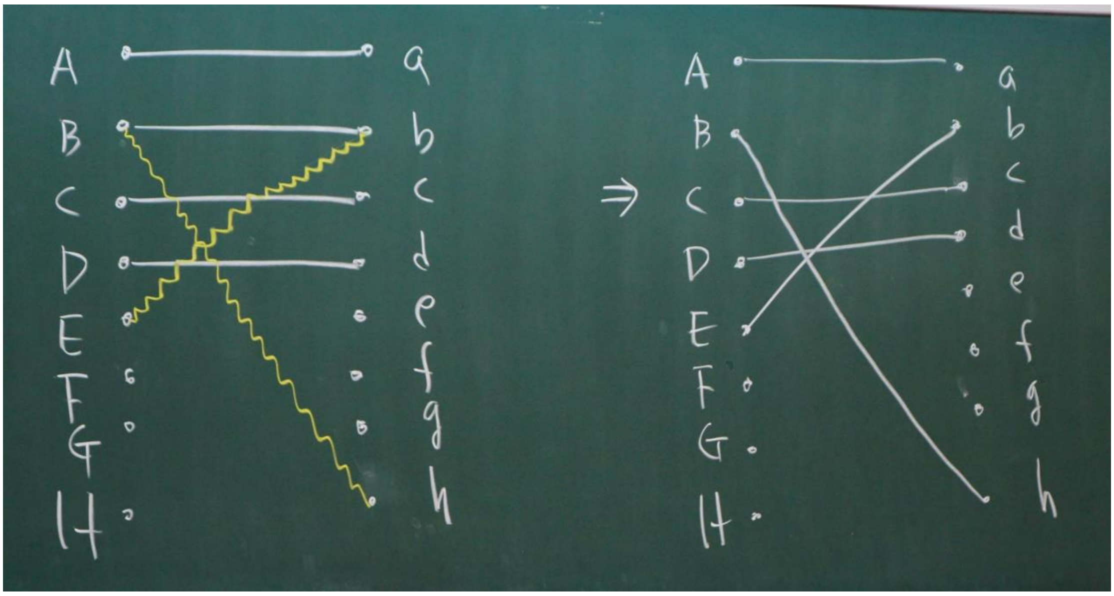
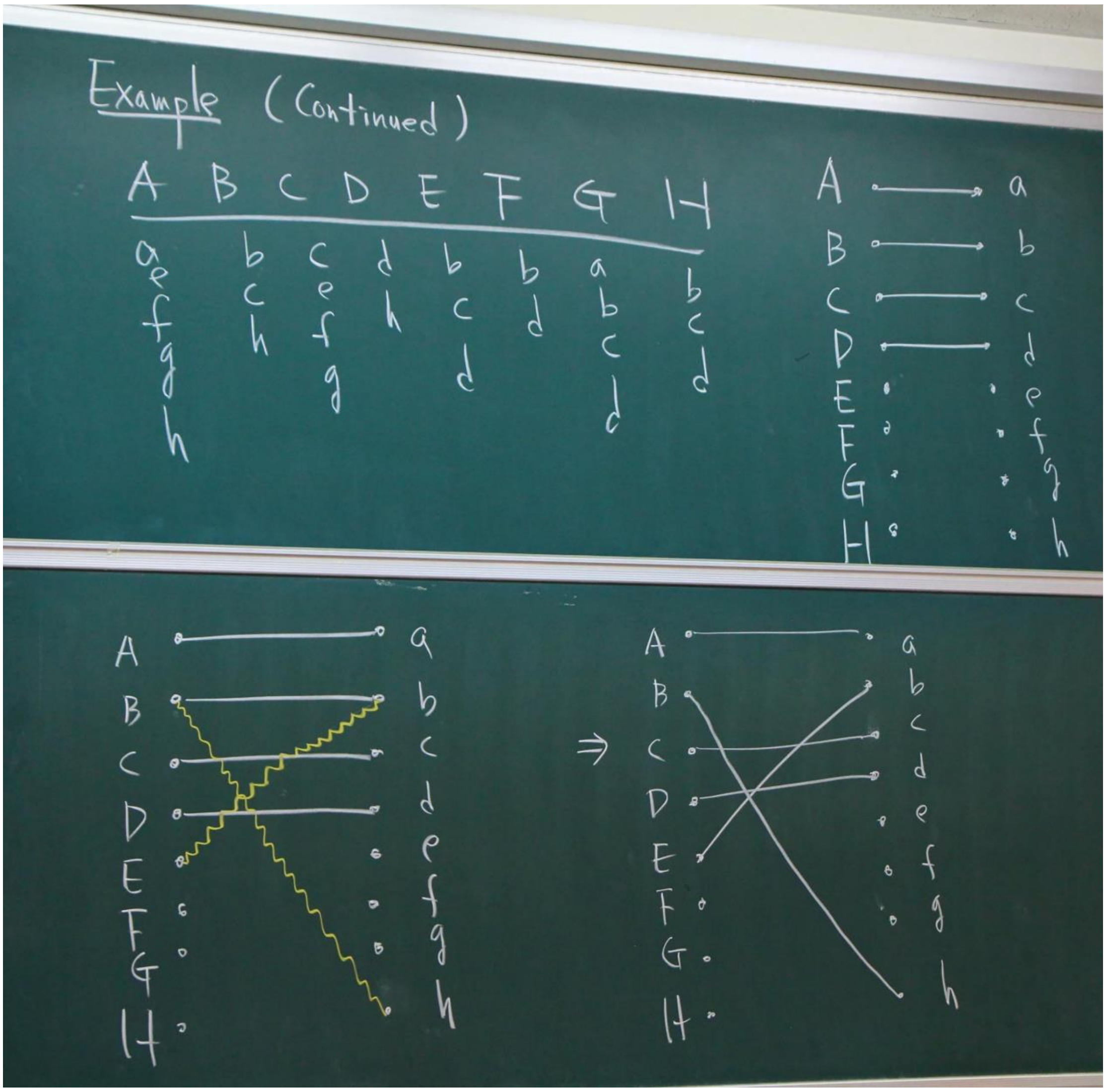
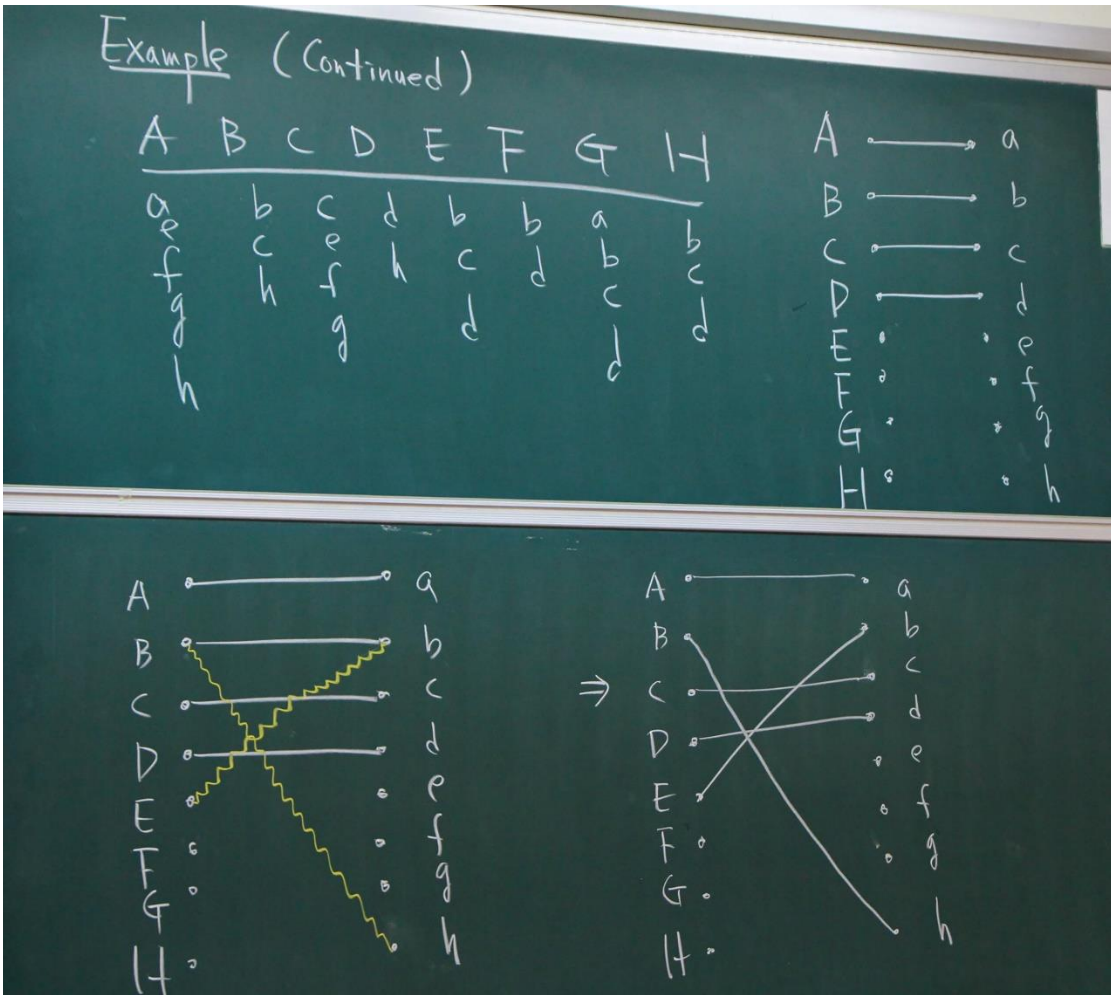
F   f
G   g
H   h


A B C D E F G H
⇒
a b c d e f g h
A B C D E F G H


Example (Continued)
A B C D E F G H
a e f g h
b c h
c e f g
d h
b c d
b d
a b c d
b c d
A B C D E F G H
a b c d e f g h
⇒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a e f g h
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b d
a b c d
b c d
A B C D E F G H
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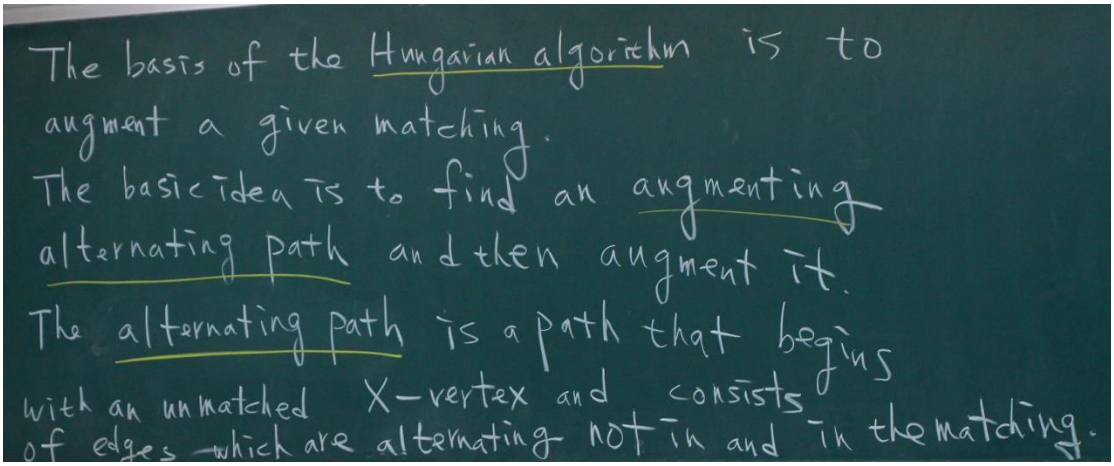
⇒


The basis of the Hungarian algorithm is to augment a given matching.
The basic idea is to find an augmenting alternating path and then augment it.
The alternating path is a path that begins with an unmatched X-vertex and consists of edges which are alternating not in and in the matching.

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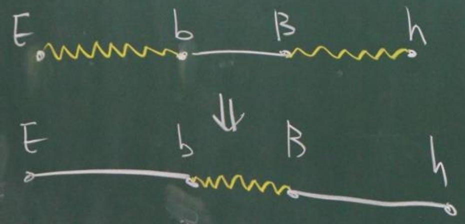
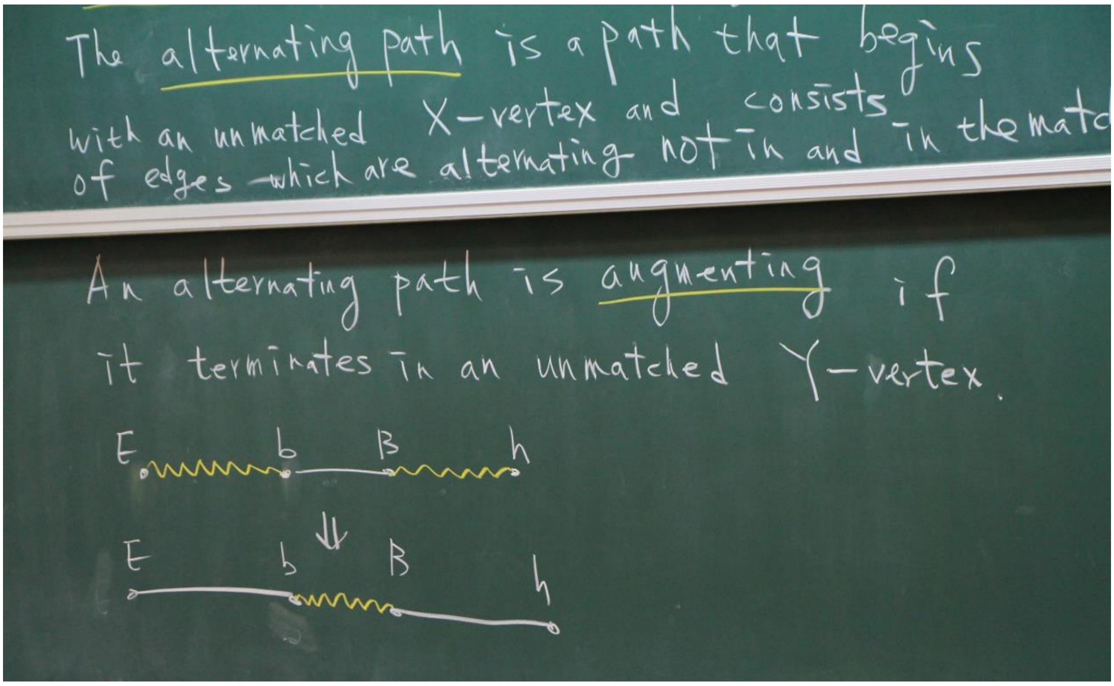
An alternating path is *augmenting* if it terminates in an unmatched Y-vertex.

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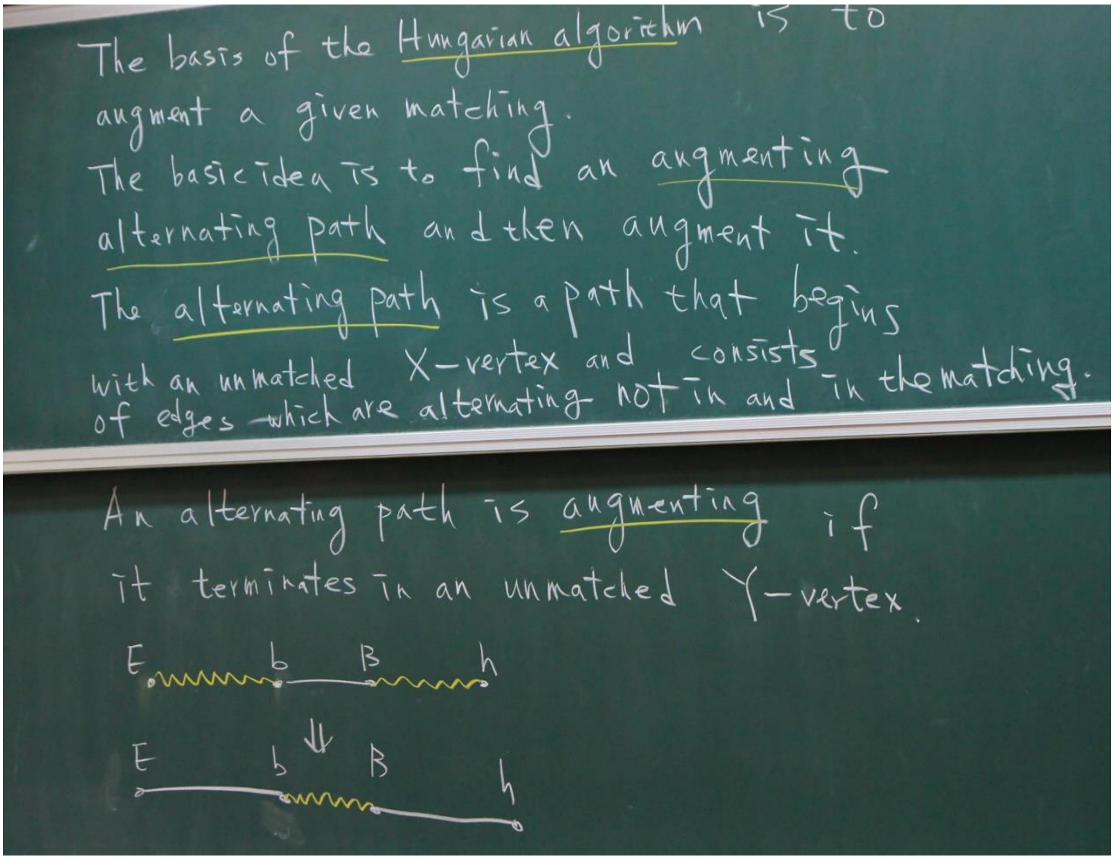
E b B h
⇓
E b B h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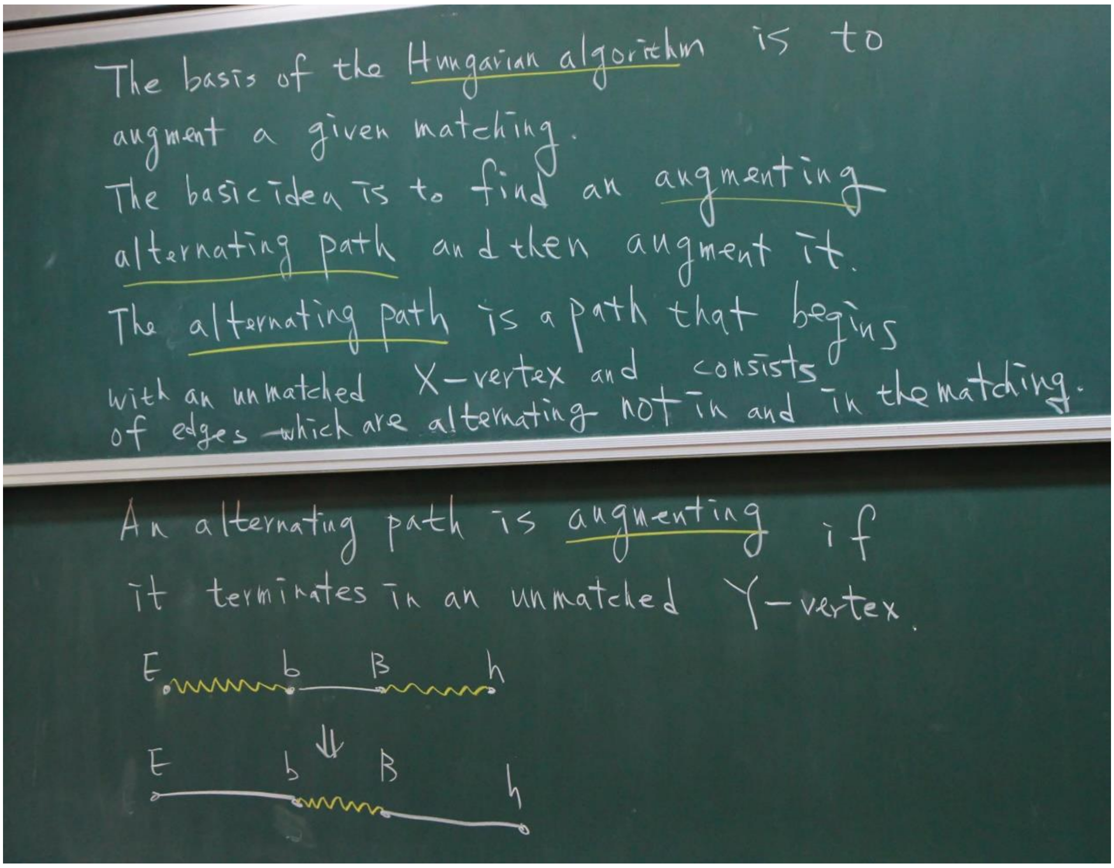
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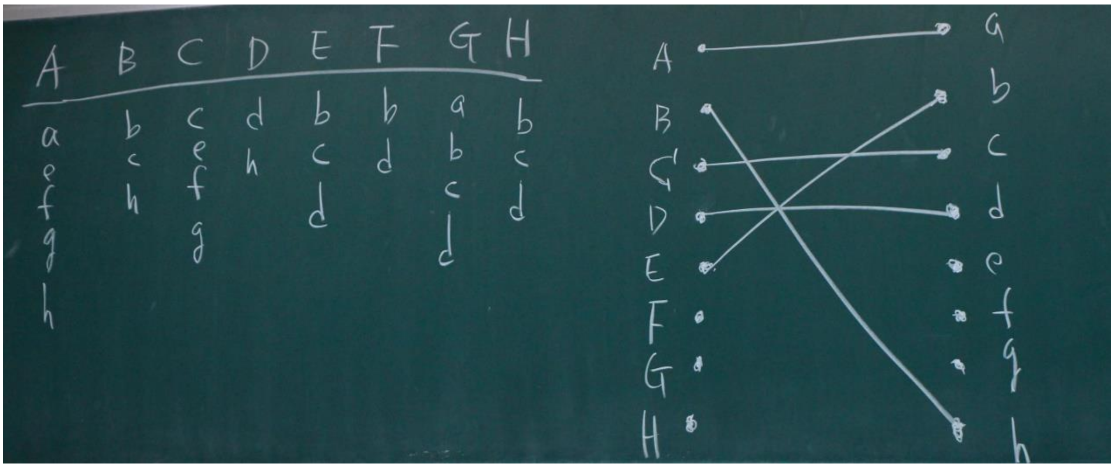
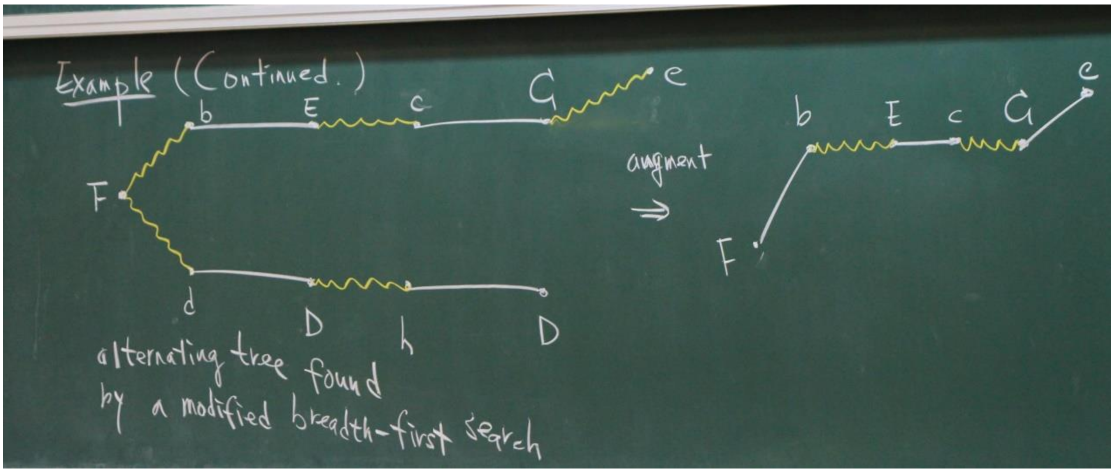
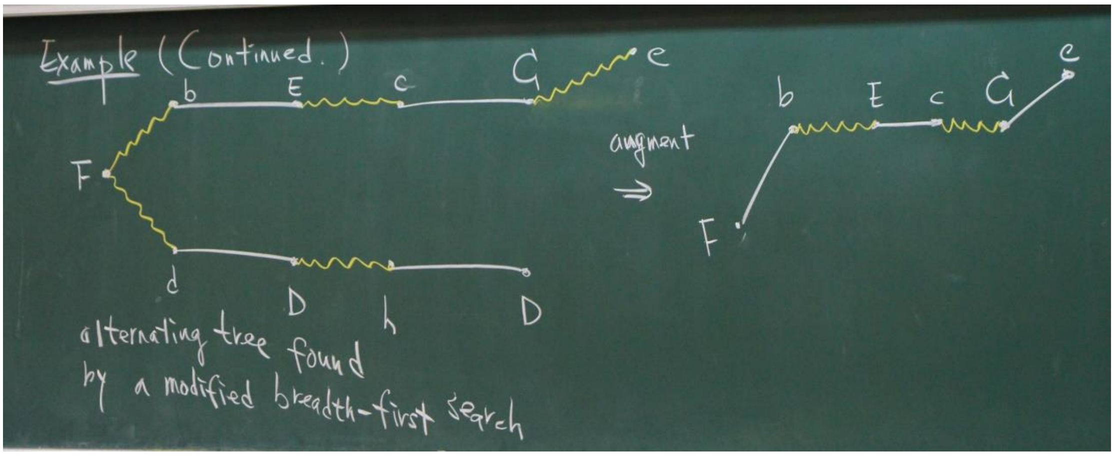

A B C D E F G H
a e f g h
b c h
c e f g
d h
b c d
b d
a b c d
b c d
A B C D E F G H
a b c d e f g h


Example (Continued.)
b
E
c
G
e
F
d
D
h
D
alternating tree found
by a modified breadth-first search
augment
⇒
b
E
c
G
e
F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b
E
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⇒
b
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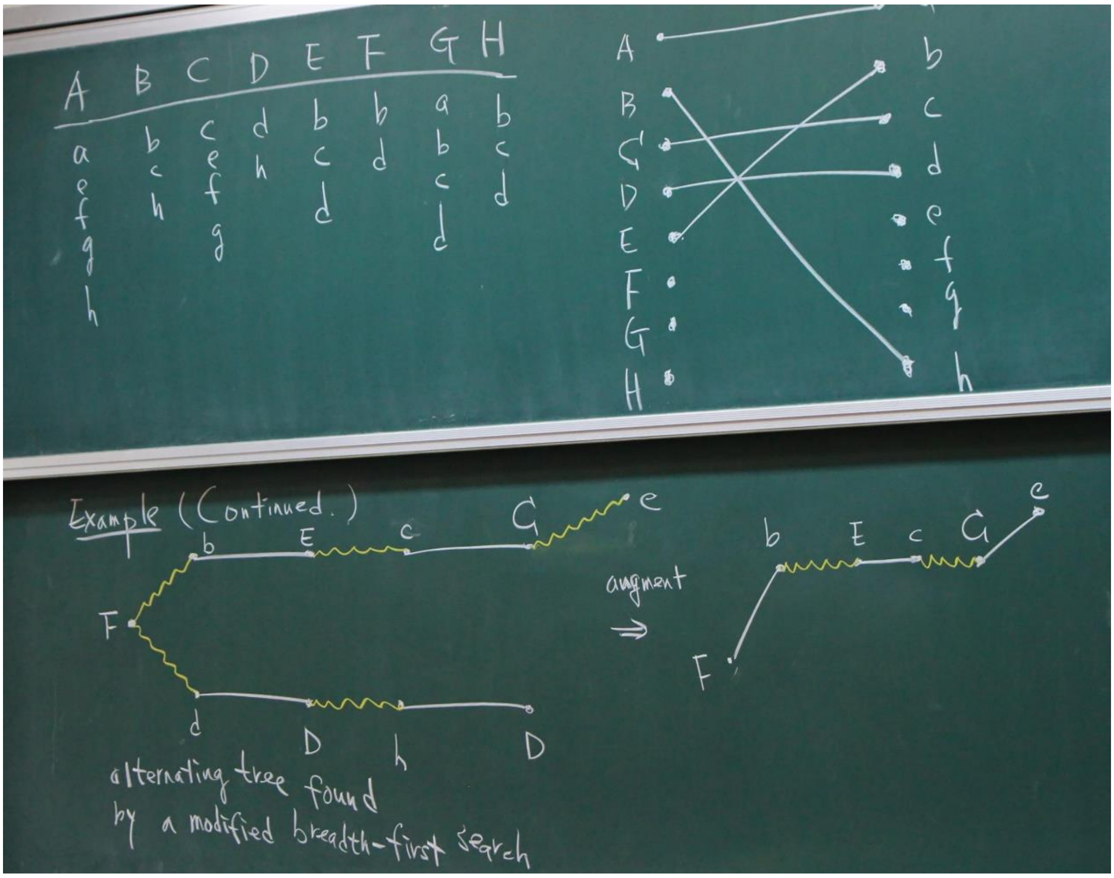

A B C D E F G H
Example (Continued.)
augment
⇒
alternating tree found
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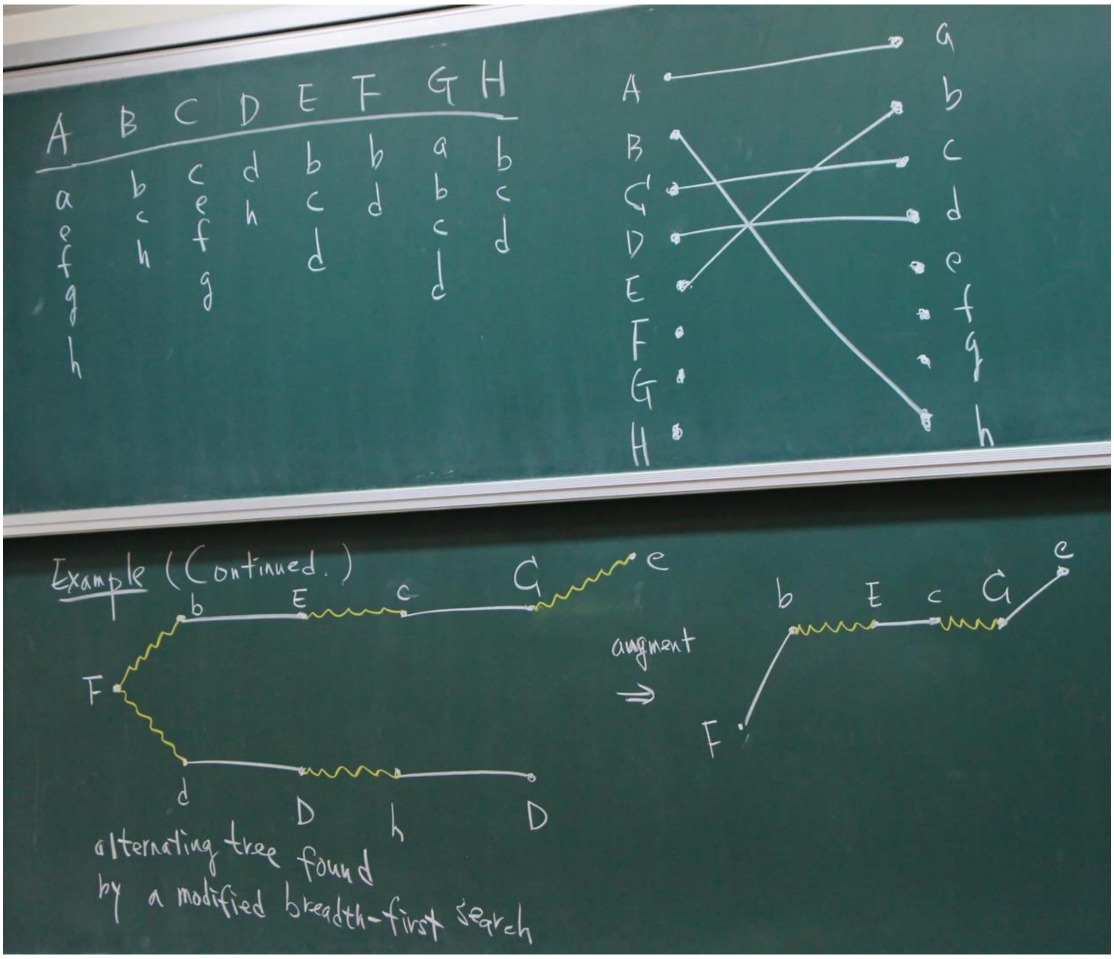
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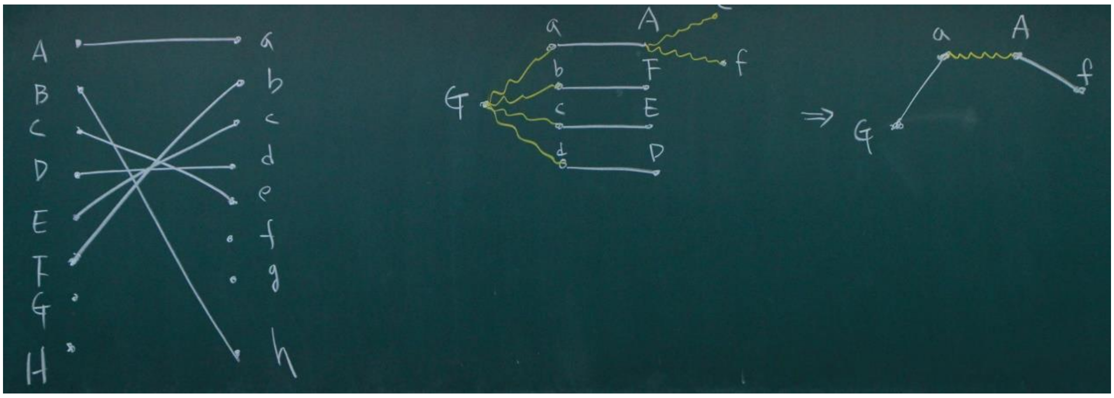
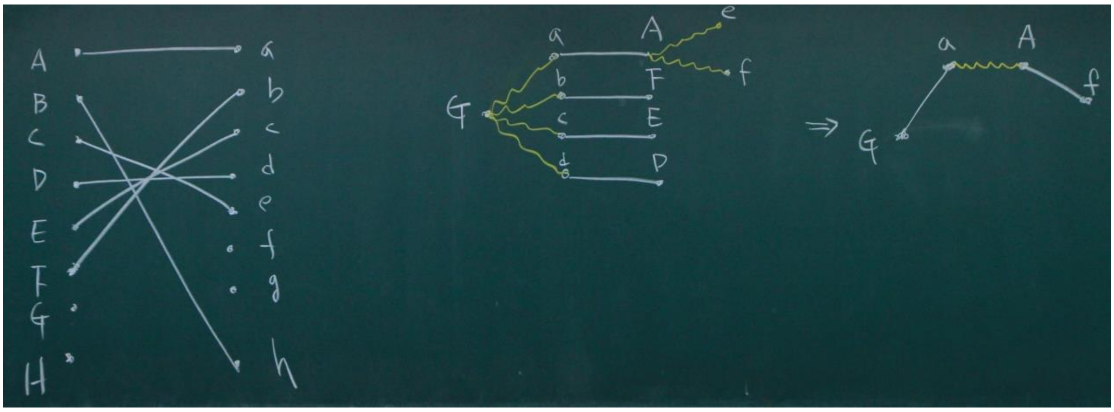
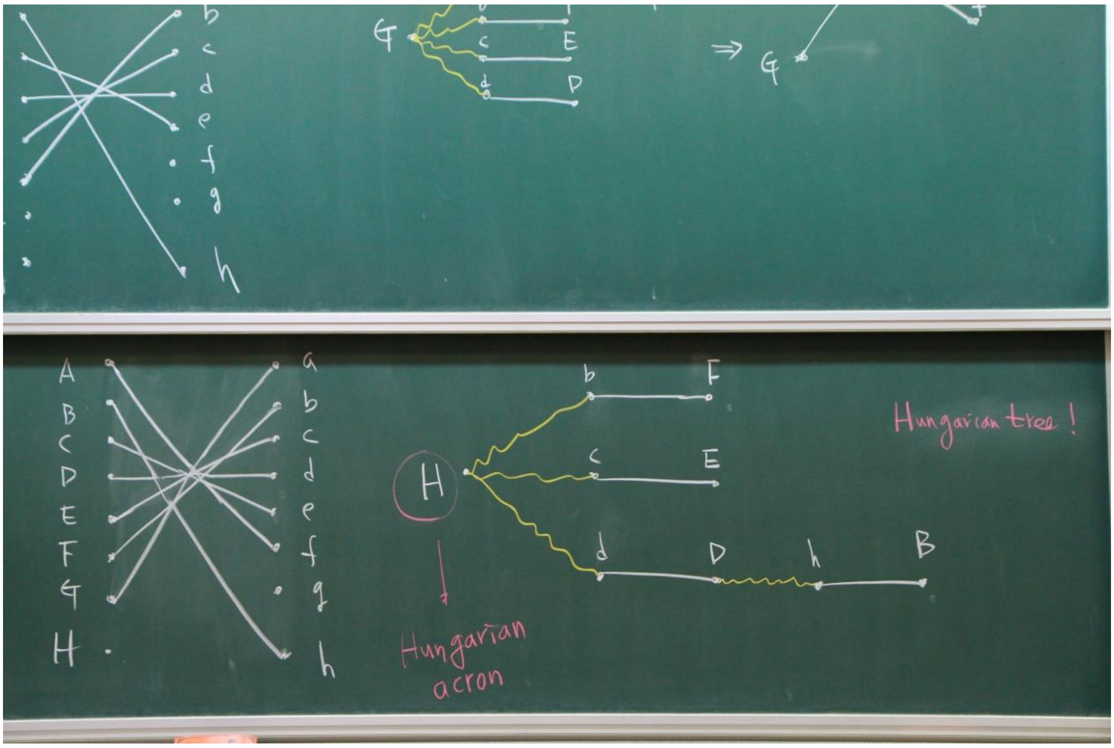
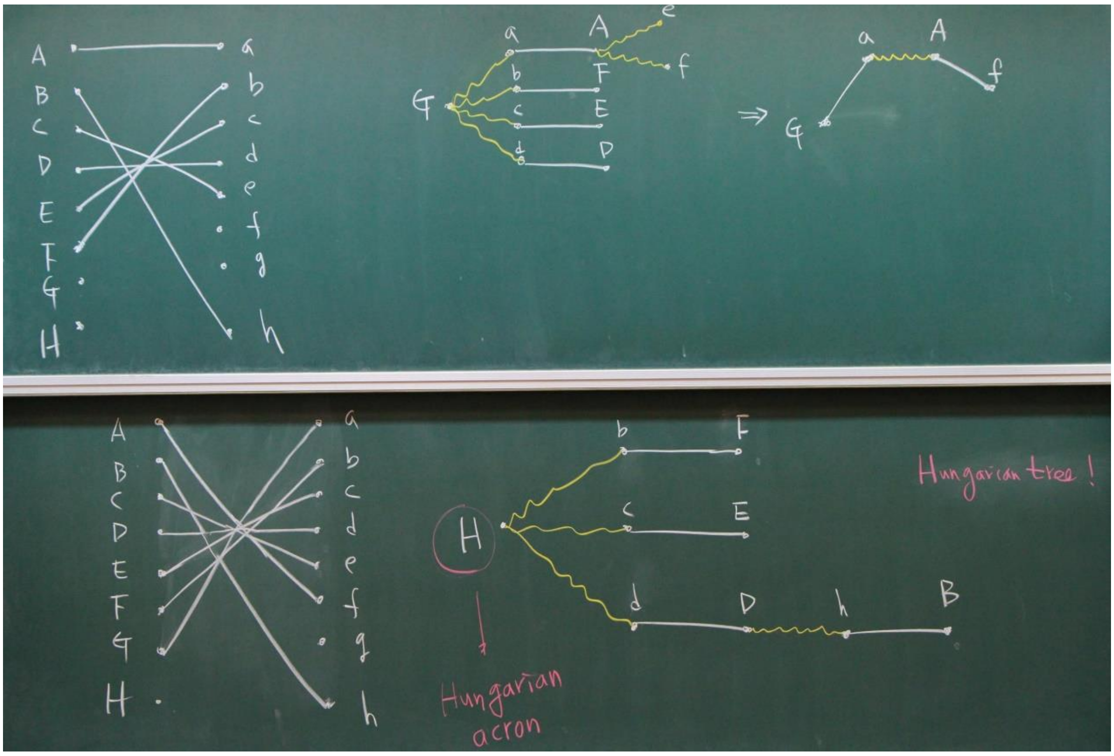





Hungarian tree!
Hungarian
acron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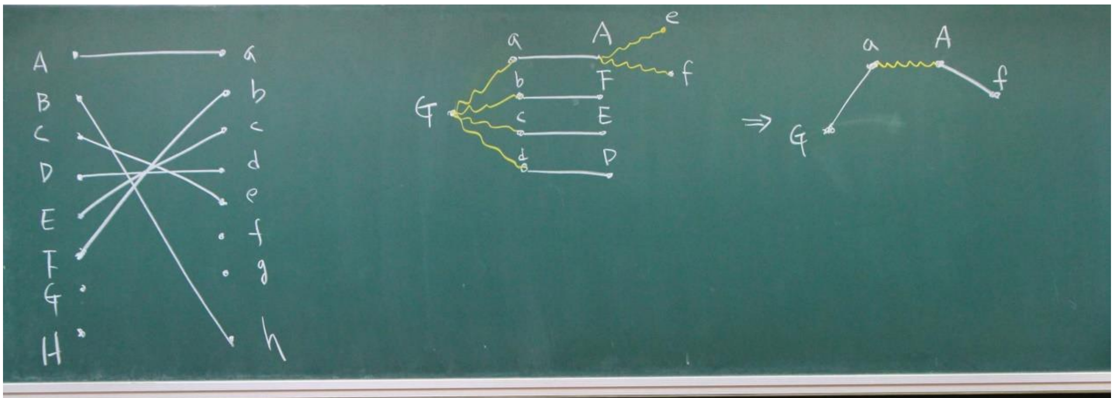

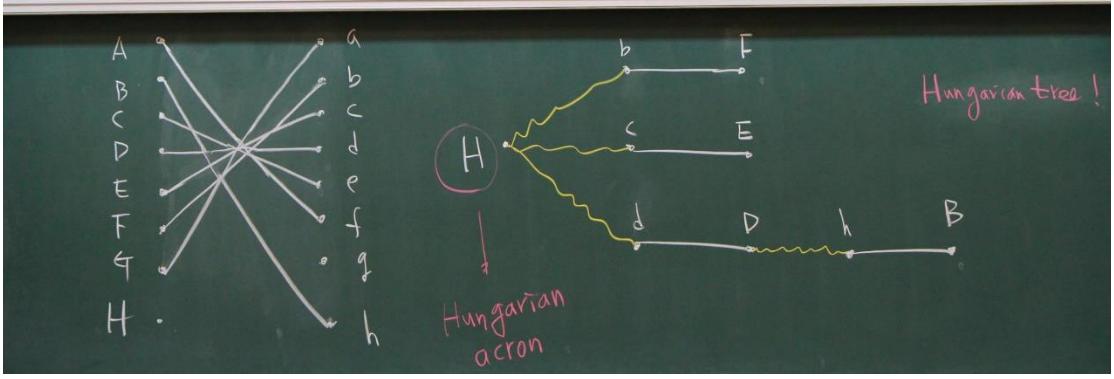

2. A matching M is a <u>maximal matching</u> if no other matching has a greater cardinality.
3. A matching M is a <u>complete matching</u> if $|M| = |X|$.

We cannot find an alternating path which is augmenting. We have arrived at a Hungarian tree.

The five girls on the tree $\{B, D, E, F, H\}$ only like the four boys $\{b, c, d, h\}$.

Hence, it is impossible for all eight girls to dance with a compatible boy.

procedure Hungarian

begin

mark all X-vertices untested

while there are unmatched, untested X-vertices

    begin

    $v :=$ an unmatched, untested X-vertex

    grow alternating tree from v

    if alternating tree is augmenting then

```
procedure Hungarian
begin
mark all X-vertices untested
while there are unmatched, untested X-vertices
    begin
    v := an unmatched, untested X-vertex
    grow alternating tree from v
    if alternating tree is augmenting then
```


```
    v := an unmatched, untested X-vertex
    grow alternating tree from v
    if alternating tree is augmenting then
```


```
        begin
        augment matching
        mark all X-vertices untested
        end
    else mark v tested
    end
end
```


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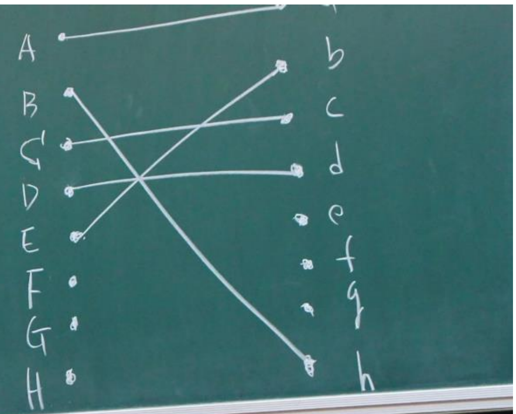
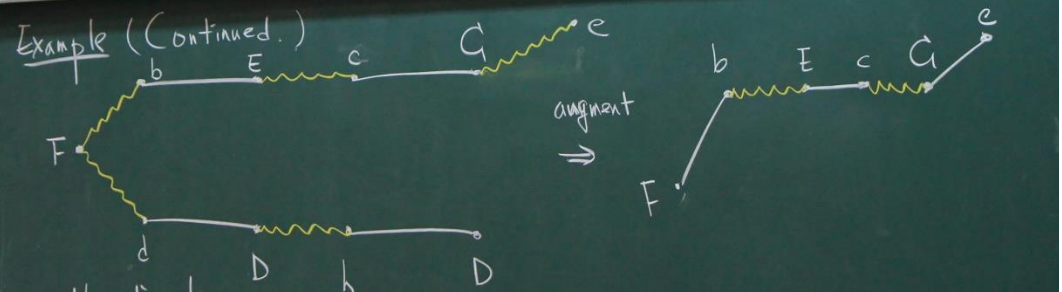


Example (Continued.)



alternating tree found by a modified breadth-first search

... of all unmatched X-vertices are Hungarian acrons.

2. When the Hungarian algorithm terminates, the resulting matching is a maximal matching.

Proof of Correctness of Hungarian Algorithm

Consider a matching (the a-matching) produced by the Hungarian algorithm. Suppose there exists a better matching (the b-matching). There must exist X-vertices which are b-matched but not a-matched. Starting with such a vertex, we can build an alternating

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Remarks

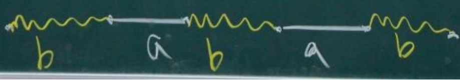
1. The Hungarian algorithm continues until either all X-vertices are matched or all unmatched X-vertices are Hungarian across.
2. When the Hungarian algorithm terminates, the resulting matching is a maximal matching.

Proof of Correctness of Hungarian Algorithm

Consider a matching (the a-matching) produced by the Hungarian algorithm. Suppose there exists a better matching (the b-matching). There must exist X-vertices which are b-matched but not a-matched. Starting with such a vertex, we can build an alternating path (b-edge, a-edge, ...).

This alternating path must end with an a-matched b-unmatched X-vertex.

Otherwise, the alternating path would be an augmenting path for the a-matching, violating the definition of the Hungarian algorithm.



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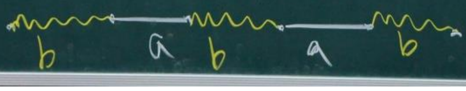

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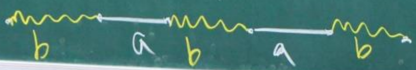
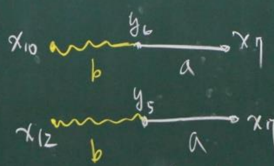
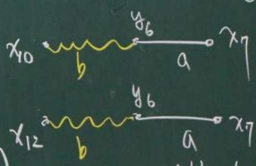


Consequently, beginning with any b-matched, a-unmatched X-vertex, we can build a alternating path which ends with a corresponding a-matched, b-unmatched X-vertex. This mapping is one-to-one. Otherwise,

e.g.,





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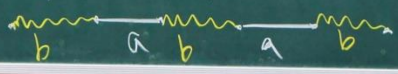

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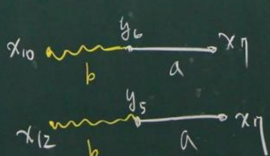

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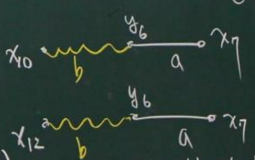

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So the b-matching does not match more X-vertices than the a-matching.